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Two-dimensional Ising model on a ruby lattice

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Abstract. We have considered a two-dimensional Ising model on a ruby lattice. The partition function is evaluated exactly by the method of Pfaffian. The Ising model on a hexagonal lattice is a special case of our model.

1. Introduction

The partition function of the two-dimensional Ising model on a square lattice was first derived by Onsager (1944). His result has been generalised to other lattices (Syozi 1972). The original derivation of Onsager is very complicated. The Pfaffian method (McCoy and Wu 1973, Montroll 1964) is probably the simplest way to solve the Ising model on a two-dimensional lattice. The purpose of this paper is to apply this method to a ruby lattice (figure 1) and evaluate the corresponding partition function exactly. Recently we have calculated the residual entropy of two-dimensional ice on this lattice (Lin and Ma 1983) and the name, ruby, was adopted for this lattice by one of us (Lin).

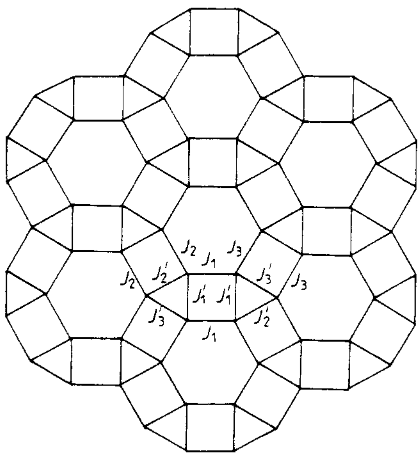

Figure 1. A ruby lattice where interaction between spins is anisotropic.

2. Ising model

The Ising model of ferromagnetism consists of a lattice of N 'spin' variables σ_i which may take on only the values +1 and −1. The energy of a lattice spin state $\{\sigma_1, \sigma_2, \ldots\}$ is

$$E = -\sum_{\mathrm{NN}} J_{ij}\sigma_i\sigma_j \tag{1}$$

where the sum is taken over all pairs i and j that are nearest neighbours (NN) in the lattice, and a periodic boundary condition is assumed. Thermodynamic properties of the lattice are obtained from the partition function

$$Z = \sum_{\sigma_1=1} \ldots \sum_{\sigma_N=1} \exp\left(\sum_{\mathrm{NN}} K_{ij}\sigma_i\sigma_j\right) \tag{2}$$

where $K = J/kT$, k is Boltzmann's constant, and T is the absolute temperature.

The Ising model on a ruby lattice is considered in this paper. We assume that the interaction between spins is anisotropic and there are six different coupling constants $(J_1, J_2, J_3, J'_1, J'_2, J'_3)$ as shown in figure 1.

The partition function can be written as (van der Waerden 1941, Newell and Montroll 1953)

$$\begin{aligned} Z &= 2^N(\cosh K_1 \cosh K_2 \cosh K_3 \cosh K'_1 \cosh K'_2 \cosh K'_3)^{N/3} \\ &\quad \times \sum n(r, s, t, u, v, w) y_1^r y_2^s y_3^t z_1^u z_2^v z_3^w, \\ y_i &= \tanh K_i, \qquad z_i = \tanh K'_i, \end{aligned} \tag{3}$$

where $n(r, s, t, u, v, w)$ is the number of closed graphs with $(r+s+t+u+v+w)$ bonds, r in the horizontal and u in the vertical direction, etc.

The partition function can be evaluated by the standard method of Pfaffian and dimer city (Kasteleyn 1963) as follows. A unit cell is shown in figure 2 which corresponds to a 24th-order matrix with elements

$$a(i, j) = -a^*(j, i). \tag{4}$$

A periodic boundary condition is assumed. The sign of each element is identified by an arrow such that its pointing from i to j implies $\operatorname{sgn} a(i, j) = +1$. A polygon with an odd number of clockwise sides is called clockwise odd. Arrows are arranged so that every closed polygon is clockwise odd. The matrix elements associated with positive signs are shown explicitly in figure 2, except those whose values are unity. For example, we have

$$a(1, 3) = 1, \qquad a(1, 8) = z_1, \qquad a(3, 5') = y_2 \exp(\mathrm{i}\phi).$$

We have

$$\begin{aligned} (1/N)\log Z &= \log 2 + \tfrac{1}{3}\log(\cosh K_1 \cosh K_2 \cosh K_3 \cosh K'_1 \cosh K'_2 \cosh K'_3) \\ &\quad + \frac{M}{2N(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \log \det A \, \mathrm{d}\theta \, \mathrm{d}\phi \end{aligned} \tag{5}$$

where $M = N/6$ is the number of unit cells in this lattice, and $\det A$ is the determinant of the matrix $a(i, j)$.

After a straightforward and long calculation, we get

$$N^{-1}\log Z = \log 2 + (48\pi^2)^{-1}\int_0^{2\pi}\int_0^{2\pi} \log F(\theta, \phi) \, \mathrm{d}\theta \, \mathrm{d}\phi \tag{6}$$

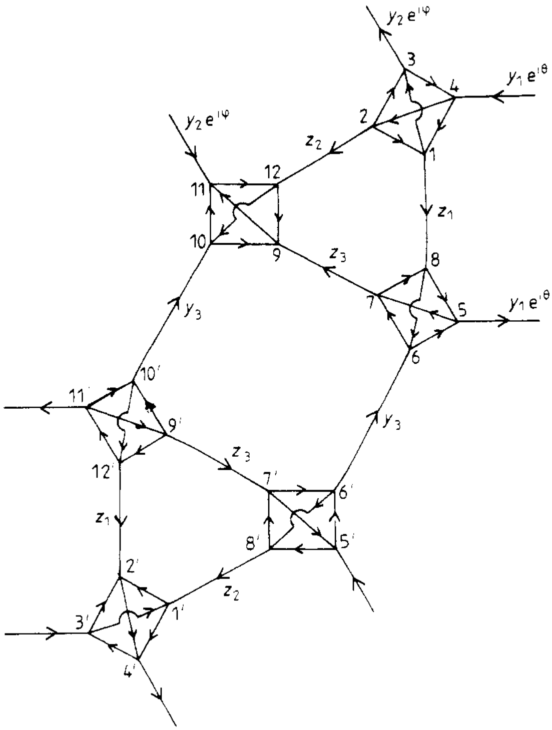

Figure 2. A unit cell of the ruby lattice which corresponds to a 24th-order matrix.

where

$$\begin{aligned}32F(\theta,\phi) = {} & a - 2b(1,2,3)\cos\phi - 2b(2,1,3)\cos\theta - 2b(3,1,2)\cos(\theta-\phi) \\ & + (S'_1S_2S_3)^2\cos 2\phi + (S_1S'_2S_3)^2\cos 2\theta + (S_1S_2S'_3)^2\cos 2(\theta-\phi) \\ & - 2S_1S_2S_3[S_1S'_2S'_3\cos(\phi-2\theta) + S'_1S_2S'_3\cos(\theta-2\phi) \\ & + S'_1S'_2S_3\cos(\theta+\phi)],\end{aligned} \tag{7}$$

$$\begin{aligned}a = {} & 2[C_1C_2C_3(C'_1C'_2C'_3 + S'_1S'_2S'_3) + C_1C'_1 + C_2C'_2 + C_3C'_3]^2 - (S'_1S_2S_3)^2 - (S_1S'_2S_3)^2 \\ & - (S_1S_2S'_3)^2 + 2(C_1S_2S_3)^2(C'_1S'_2S'_3 + S'_1C'_2C'_3)^2 \\ & + 2(S_1C_2S_3)^2(S'_1C'_2S'_3 + C'_1S'_2C'_3)^2 \\ & + 2(S_1S_2C_3)^2(S'_1S'_2C'_3 + C'_1C'_2S'_3)^2,\end{aligned}$$

$$\begin{aligned}b(i,j,k) = {} & 2(C_1C'_1 + C_2C'_2 + C_3C'_3)C_iS_jS_k(C'_iS'_jS'_k + S'_iC'_jC'_k) \\ & + 2C_1C_2C_3C_iS_jS_k(C'_1C'_2C'_3 + S'_1S'_2S'_3)(C'_iS'_jS'_k + S'_iC'_jC'_k) \\ & - 2S_i^2S_jC_jS_kC_k(S'_iC'_jS'_k + C'_iS'_jC'_k)(S'_iS'_jC'_k + C'_iC'_jS'_k) - S_1S_2S_3S_iS'_jS'_k,\end{aligned}$$

$$S_i \equiv \sinh 2K_i, \qquad S'_i \equiv \sinh 2K'_i,$$

$$C_i \equiv \cosh 2K_i, \qquad C'_i \equiv \cosh 2K'_i.$$

It can be shown that

$$F(\theta,\phi) \geqslant F(0,0) = P^2/16 \tag{8}$$

where

$$P \equiv C_1C'_1 + C_2C'_2 + C_3C'_3 + C_1C_2C_3(C'_1C'_2C'_3 + S'_1S'_2S'_3)$$
$$- C_1S_2S_3(C'_1S'_2S'_3 + S'_1C'_2C'_3)$$
$$- S_1C_2S_3(S'_1C'_2S'_3 + C'_1S'_2C'_3) - S_1S_2C_3(S'_1S'_2C'_3 + C'_1C'_2S'_3)$$

and the equality holds if and only if $\theta = \phi = 0$. Therefore the critical temperature T_c is determined by $P = 0$.

When J_i (or J'_i) $\to \infty$, equation (6) agrees with the known result for the triangular (or hexagonal) lattice. In the general case, if the star-triangle transformation (Syozi 1972) is applied to each triangle of the ruby lattice, we get a non-uniform hexagonal lattice which consists of two different kinds of hexagons as far as interaction between spins is concerned (figure 3).

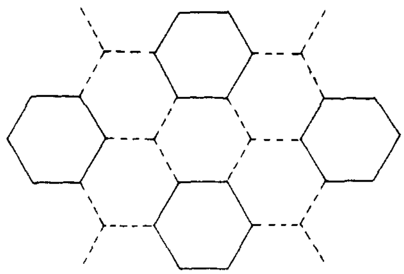

Figure 3. The non-uniform hexagonal lattice obtained from the ruby lattice by star-triangle transformation.

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